Name of College :- S. S. College J-Bad

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Deptt - Mathematics Class - B.Sc(I) HONS

Topic - Problem Based on Partial Differentiation (Differential Calculus)

Period - I + II

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Problem based on the Verification of Euler's theorem —

1. Verify Euler's theorem when

$$u = \frac{x(x^3-y^3)}{x^3+y^3}$$

Solution →

$$u = \frac{x(x^3-y^3)}{x^3+y^3}$$

Taking log on both sides.

$$\log u = \log x + \log(x^3-y^3) - \log(x^3+y^3) \quad \text{--- (I)}$$

Differentiating (I) partially with respect to x

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{1}{x^3-y^3}\cdot 3x^2 - \frac{1}{x^3+y^3}\cdot 3x^2$$

$$\frac{1}{u}x\frac{\partial u}{\partial x} = 1 + \frac{3x^3}{x^3-y^3} - \frac{3x^3}{x^3+y^3} \quad \text{--- (II)}$$

Again differentiating (I) partially with respect to y

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{1}{x^3-y^3}(-3y^2) - \frac{1}{x^3+y^3}\cdot 3y^2 \quad \text{--- (III)}$$

$$\Rightarrow \frac{1}{u}y\frac{\partial u}{\partial y} = \frac{-3y^3}{x^3-y^3} - \frac{3y^3}{x^3+y^3}$$

Now $\frac{1}{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=1+\frac{3(x^3-y^3)}{x^3-y^3}-\frac{3(x^3+y^3)}{x^3+y^3}$

$= 1+3-3$

$= 1$

$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u$

Since the given function is a homogeneous function of degree 1. therefore u satisfies Euler's theorem.

2. Verify Euler's theorem when $u = x^3 \log y/x$

Solution → By the question

$u = x^3 \log y/x$ — (1)

it is homogeneous function of degree 3

then we have to show

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u$

for $u = x^3 \log y/x$

$= x^3(\log y - \log x)$

$= x^3 \log y - x^3 \log x$

$\therefore \frac{\partial u}{\partial x} = 3x^2\log y - \left[3x^2\log x + x^3\cdot\frac{1}{x}\right]$

$= 3x^2\log y - 3x^2\log x - x^2$

$\therefore x\frac{\partial u}{\partial x} = 3x^3\log y - 3x^3\log x - x^3$

$= 3x^3(\log y - \log x) - x^3$

Also $\cancel{\dfrac{\partial \phi}{\partial y} = \dfrac{x^3}{y} \cdot \dfrac{\partial \phi}{\partial y}}$

$\cancel{y\dfrac{\partial \phi}{\partial y} = x^3 \dfrac{\partial \phi}{\partial y}}$ $\quad \cancel{\dfrac{\partial \phi}{\partial y} = \dfrac{x^3}{y^2}}$

$$\frac{\partial \phi}{\partial y} = \frac{x^3}{y}$$

$$y\frac{\partial \phi}{\partial y} = x^3$$

$$\therefore \; x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} = 3x^3(\log y - \log x) - \cancel{x^3} + \cancel{x^3}$$
$$= 3x^3(\log y - \log x)$$
$$= 3\left[x^3(\log y/x)\right]$$
$$= 3\phi$$

Thus Euler's theorem is verified

Problem 2.

Verify Euler's theorem when

$$u = x^n \sin\frac{y}{x}$$

Solution - Let $u = x^n \sin\dfrac{y}{x}$ — (1)

Clearly this function is homogeneous function of degree n.

Then we are to show

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

for

Diff. (1) partially with respect to x,

$$\frac{\partial u}{\partial x} = nx^{n-1}\sin y/x + x^n \cos\frac{y}{x} \cdot \left(-\frac{1}{x^2}\right) \cdot y$$
$$= nx^{n-1}\sin y/x - yx^{n-2}\cos y/x$$

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$$\therefore x\frac{\partial u}{\partial x} = n x^n \sin y/x - x^{n-1} y \cos y/x$$

Again diff. (1) partially w.r.t.
respect to y

$$\frac{\partial u}{\partial y} = x^n \cos\frac{y}{x} \cdot \frac{1}{x}$$

$$= x^{n-1} \cos y/x$$

$$y\frac{\partial u}{\partial y} = x^{n-1} y \cos y/x$$

Thus $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n x^{n} \sin\frac{y}{x} - \cancel{x^{n-1} y \cos\frac{y}{x}} + \cancel{x^{n-1} y \cos y/x}$$

$$= n x^n \sin y/x$$

$$= n(x^n \sin y/x)$$

$$= nu$$

Thus Euler's theorem is verified.

If $P\,dx + Q\,dy + R\,dz$ can be made a perfect differential of some function of x, y, z by multiplying each term by a common factor, prove that -

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

Solution —

Let-

$du = \lambda(P\,dx + Q\,dy + R\,dz)$

Where λ is the function of x, y, z and $\lambda \neq 0$.

But $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$\therefore \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = P\lambda\,dx + Q\lambda\,dy + R\lambda\,dz$

then On Equaling Co-efficients...

$\frac{\partial u}{\partial x} = P\lambda$ ——— (I)

$\frac{\partial u}{\partial y} = Q\lambda$ ——— (II)

$\frac{\partial u}{\partial y} = R\lambda$ ——— (III)

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(P\lambda)$

or $\frac{\partial^2 u}{\partial y \partial x} = P\frac{\partial \lambda}{\partial y} + \lambda\frac{\partial P}{\partial y}$

Again $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(Q\lambda) = \lambda\frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$

$\Rightarrow \frac{\partial^2 u}{\partial x \partial y} = Q\frac{\partial \lambda}{\partial x} + \lambda\frac{\partial Q}{\partial x}$

Since $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\Rightarrow P\frac{\partial \lambda}{\partial y} + \lambda\frac{\partial P}{\partial y} = Q\frac{\partial \lambda}{\partial x} + \lambda\frac{\partial Q}{\partial x}$

$$\Rightarrow x\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial x}{\partial x} - P\frac{\partial x}{\partial y}$$

$$\Rightarrow Rx\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = QR\frac{\partial x}{\partial x} - PR\frac{\partial x}{\partial y} \quad \text{—— (iv)}$$

Similarly from the relations

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 u}{\partial z \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial^2 u}{\partial x \partial z}$$

$$\Rightarrow Px\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = RP\frac{\partial x}{\partial y} - PQ\frac{\partial x}{\partial z} \quad \text{—— (v)}$$

$$\text{and } Qx\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = PQ\frac{\partial x}{\partial z} - QR\frac{\partial x}{\partial x} \quad \text{—— (vi)}$$

Adding (iv), (v) and (vi) and dividing by x, we get—

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

4. If A, B, C be the angles of a triangle such that

$$\sin^2 A + \sin^2 B + \sin^2 C = \text{const.}$$

prove that—

$$\frac{dA}{dB} = \frac{\tan C - \tan B}{\tan A - \tan C}$$

Solution 1 →

Since

$$A+B+C=\pi$$

$$\Rightarrow \frac{dA}{dB}+1+\frac{dC}{dB}=0$$

$$\frac{dC}{dB}=-\left(1+\frac{dA}{dB}\right) \quad \text{——— (1)}$$

By question

$$\sin^2 A+\sin^2 B+\sin^2 C=\text{Const.}$$

Diff. with respect to B.

$$\frac{d\sin^2 A}{d\sin A}\cdot\frac{d\sin A}{dA}\cdot\frac{dA}{dB}+\frac{d\sin^2 B}{d\sin B}\cdot\frac{d\sin B}{dB}+\frac{d\sin^2 C}{d\sin C}\cdot\frac{d\sin C}{dC}\cdot\frac{dC}{dB}=0$$

$$\Rightarrow 2\sin A\cos A\frac{dA}{dB}+2\sin B\cos B+2\sin C\cos C\frac{dC}{dB}=0$$

$$\Rightarrow \sin 2A\frac{dA}{dB}+\sin 2B+\sin 2C\frac{dC}{dB}=0$$

using (1)

$$\sin 2A\frac{dA}{dB}+\sin 2B-\sin 2C\left(1+\frac{dA}{dB}\right)=0$$

or $\frac{dA}{dB}(\sin 2A-\sin 2C)=\sin 2C-\sin 2B$

$$\Rightarrow \frac{dA}{dB}\left[2\cos(A+C)\sin(A-C)\right]=2\cos(C+B)\sin(C-B)$$

$$\Rightarrow \frac{dA}{dB}\left[\cos(A+C)\sin(A-C)\right]=\cos(B+C)\sin(C-B)$$

$$\Rightarrow \frac{dA}{dB}\cos(\pi-B)\left[\sin A\cos C-\cos A\sin C\right]=\cos(\pi-A)\left[\sin C\cos B-\cos C\sin B\right]$$

$$\Rightarrow -\frac{dA}{dB}\cos B\left[\sin A\cos C-\cos A\sin C\right]=-\cos A\left[\sin C\cos B-\cos C\sin B\right]$$

$$\Rightarrow -\frac{dA}{dB}\cos A\cos B\cos C\left[\tan A-\tan C\right]=-\cos A\cos B\cos C\left[\tan C-\tan B\right]$$

$$\Rightarrow \frac{dA}{dB}\left(\tan A-\tan C\right)=\tan C-\tan B$$

$$\therefore \frac{dA}{dB}=\frac{\tan C-\tan B}{\tan A-\tan C}$$ Proved

Problem 5. If,

$$u=F(x-y,\ y-z,\ z-x)$$

and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ all exist,

Prove that -

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Solution

We have

$$u=F(x-y,\ y-z,\ z-x)$$

Let $x-y=X \Rightarrow \frac{\partial X}{\partial x}=1 \quad \frac{\partial X}{\partial y}=-1 \quad \frac{\partial X}{\partial z}=0$

$y-z=Y \quad \frac{\partial Y}{\partial x}=0 \quad \frac{\partial Y}{\partial y}=1 \quad \frac{\partial Y}{\partial z}=-1$

$z-x=Z. \quad \frac{\partial Z}{\partial x}=-1 \quad \frac{\partial Z}{\partial y}=0 \quad \frac{\partial Z}{\partial z}=1$

$\therefore$ (1) Transform in to

$$u=F(X,Y,Z)$$

$$\frac{\partial u}{\partial x}=\frac{\partial F}{\partial X}\cdot\frac{\partial X}{\partial x}+\frac{\partial F}{\partial Y}\cdot\frac{\partial Y}{\partial x}+\frac{\partial F}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial F}{\partial X}\cdot 1 + \frac{\partial F}{\partial Y}\,0 + \frac{\partial F}{\partial Z}(-1)$$

$$= \frac{\partial F}{\partial X} - \frac{\partial F}{\partial Z}$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial F}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial F}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial F}{\partial Z}\cdot\frac{\partial Z}{\partial y}$$

$$= -\frac{\partial F}{\partial X} + \frac{\partial F}{\partial Y}$$

And $\dfrac{\partial u}{\partial z} = \dfrac{\partial F}{\partial X}\cdot\dfrac{\partial X}{\partial z} + \dfrac{\partial F}{\partial Y}\cdot\dfrac{\partial Y}{\partial z} + \dfrac{\partial F}{\partial Z}\cdot\dfrac{\partial Z}{\partial z}$

$$= 0 - \frac{\partial F}{\partial Y} + \frac{\partial F}{\partial Z}$$

$$= \frac{\partial F}{\partial Z} - \frac{\partial F}{\partial Y}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{\frac{\partial F}{\partial X}} - \cancel{\frac{\partial F}{\partial Z}} - \cancel{\frac{\partial F}{\partial X}} + \cancel{\frac{\partial F}{\partial Y}} + \cancel{\frac{\partial F}{\partial Z}} - \cancel{\frac{\partial F}{\partial Y}}$$

$$= 0$$

hence the result.